Difference Sturm–Liouville problems in the imaginary direction

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We consider difference operators in L^2 on $\mathbb{R}$ of the form

$$\mathcal{L}f(s) = p(s)f(s+i) + q(s)f(s) + r(s)f(s-i),$$

where i is the imaginary unit. The domain of definiteness are functions holomorphic in a strip with some conditions of decreasing at infinity. Problems of such type with discrete spectra are well known (Meixner–Pollaszek, continuous Hahn, continuous dual Hahn, and Wilson hypergeometric orthogonal polynomials). We write explicit spectral decompositions for several operators $\mathcal{L}$ with continuous spectra. We also discuss analogs of 'boundary conditions' for such operators.

1 Introduction

1.1. Formulation of problem. Consider the space L^2 on $\mathbb{R}$ with respect to a positive weight $w(s)\,ds$. Consider a subspace H consisting of functions $f(s)$ holomorphic in the strip $-1 < \operatorname{Im} s < 1$ smooth up to the boundary $\operatorname{Im} s = \pm 1$ and sufficiently rapidly decreasing in the strip as $|s| \to \infty$. We consider difference operators in $L^2(\mathbb{R}, w(s)\,ds)$ of the form

$$\mathcal{L}f(s) = p(s)f(s+i) + q(s)f(s) + r(s)f(s-i),$$

where i is the imaginary unit; the domain of definiteness of $\mathcal{L}$ is the subspace H. For such operators we discuss essential self-adjointness and the eigenvalue problem

$$\mathcal{L}f(s) = \lambda f(s).$$

Our main purpose is spectral decomposition. In fact, several problems of this kind were solved (see the list below). All solved problems had the following form. Denote

$$\mu(s) = e^{cs}\frac{\prod_{k=1}^{m}\Gamma(a_k + is)}{\prod_{l=1}^{n}\Gamma(b_l + is)}, \tag{1.1}$$

where $c \in \mathbb{R}$, and

$$\nu(s) = \overline{\mu(\overline{s})} = e^{cs}\frac{\prod_{k=1}^{m}\Gamma(\overline{a}_k - is)}{\prod_{l=1}^{n}\Gamma(\overline{b}_l - is)}. \tag{1.2}$$

Denote

$$A(s) := \frac{\nu(s+i)}{\nu(s)} = e^{-ic}\frac{\prod_{k=1}^{m}(\overline{a}_k - is)}{\prod_{l=1}^{n}(\overline{b}_l - is)}, \tag{1.3}$$

$$B(s) := \frac{\mu(s-i)}{\mu(s)} = e^{ic}\frac{\prod_{k=1}^{m}(a_k + is)}{\prod_{l=1}^{n}(b_l + is)}. \tag{1.4}$$

We consider the space $L^2(\mathbb{R}, w(s)\, ds)$ with respect to the weight

$$w(s)\, ds := \frac{1}{2\pi}\mu(s)\nu(s)$$

and the difference operator

$$\mathcal{L}f(s) = A(s)f(s+i) - \big(A(s) + B(s)\big)f(s) + B(s)f(s-i). \qquad (1.5)$$

1.2. Neo-classical orthogonal polynomials. Now we enumerate solved problems of this kind. We use the standard notation for hypergeometric functions

$$_pF_q\begin{bmatrix} a_1, \ldots, a_q \\ b_1, \ldots, b_p \end{bmatrix}; z \end{bmatrix} := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n\, z^n}{(b_1)_n \ldots (b_q)_n\, n!},$$

where $(a)_n := a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol.

Recall that there are 3 types of classical hypergeometric orthogonal polynomials, see [3], [13], [12]. Polynomials of the first type are solutions of the usual Sturm–Liouville problems for second order differential operators: Jacobi (including Gegenbauer, Legendre, Chebyshev), Laguerre, Hermite systems (see [9]).

Polynomials of the second type are solutions of difference Sturm–Liouville problem on lattices: Racah, (Chebyshev)–Hahn, dual Hahn, Meixner, Krawtchouk, Charlier, see [25], [13], [12].

Polynomials of the third type are solutions of Sturm–Liouville problems of the form (1.1)–(1.5): Wilson, continuous Hahn, continuous dual Hahn, Meixner–Pollaczek systems, see [13], [1]. Recall that all classical polynomial orthogonal systems are degenerations of the Wilson polynomials, see [3], [13], [12].

a) *The Meixner–Pollaczek system* or *Meixner polynomials of the second kind*, see [19], [13], Section 1.7. We take

$$\mu(s) = e^{(\varphi - \pi/2)s}\Gamma(a + is),$$

where parameters a, φ satisfy $a > 0$, $0 < \varphi < \pi$. Therefore

$$w(s) = \frac{1}{2\pi}e^{(2\varphi - \pi)s}\Gamma(a + is)\Gamma(a - is). \qquad (1.6)$$

and the difference operator is

$$\mathcal{L}f(s) = ie^{-i\varphi}(a - is)f(s+i) + 2(-s\cos\varphi + \lambda\sin\varphi)f(s)-$$
$$= ie^{i\varphi}(a + is)f(s-i). \qquad (1.7)$$

The eigenfunctions are polynomials

$$P_n(s) = \frac{(2a)_n}{n!}e^{in\varphi}\,_2F_1\begin{bmatrix} -n, a + is \\ 2a \end{bmatrix}; 1 - e^{-2i\varphi} \end{bmatrix},$$
$$\mathcal{L}P_n(s) = n\sin\varphi\, P_n(s).$$

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Norms of Meixner–Pollaczek polynomials are given by

$$\|w(s)\|^2 := \int_{-\infty}^{\infty} |p_n(s)|^2 w(s)\, ds = \frac{\Gamma(n+2a)}{(2\sin\varphi)\, n!}.$$

Recall (see [8], formula 1.18(6)) that

$$|\Gamma(a+is)| \sim \sqrt{2\pi}|s|^{a-1/2}e^{-\pi s/2}, \qquad s \to \infty. \tag{1.8}$$

Therefore the weight $w(s)$ exponentially decreases and the space $L^2(\mathbb{R}, w(s)\, ds)$ contains all polynomials. The operator $\mathcal{L}$ send a polynomial to a polynomial of the same degree, therefore our Sturm–Liouville problem is pure algebraic. The same remarks hold for 3 polynomial systems discussed below.

b) *The continuous Hahn system,* see [6], [2], [25], [13]. In this case,

$$\mu(s) = \Gamma(a+is)\Gamma(b+is),$$

where the parameters a, b satisfy $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$. The eigenfunctions are polynomials

$$p_n(s) := i^n \frac{(a+\overline{a})_n(a+\overline{b})_n}{n!} \, {}_3F_2\left[\begin{matrix} -n, n+a+b+\overline{a}+\overline{b}, a+is \\ a+\overline{a}, a+\overline{b} \end{matrix}; 1\right],$$

$$\mathcal{L}p_n = n(n+a+\overline{a}+b+\overline{b})p_n.$$

c) *The continuous dual Hahn system,* see [35], [13], [12]. In this case

$$\mu(s) = \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)},$$

where the parameters a, b, c satisfy $a > 0$, $b > 0$, $c > 0$ or $a > 0$, $\operatorname{Re} b > 0$, $c = \overline{b}$. We consider *even* orthogonal polynomials $p_n(s^2)$:

$$p_n(s^2) := (a+b)_n(a+c)_n \, {}_3F_2\left[\begin{matrix} -n, a+is, a-is \\ a+b, a+c \end{matrix}; 1\right],$$

$$\mathcal{L}p_n = np_n.$$

d) *Wilson system,* see [35], [1], [13], [21]. In this case,

$$\mu(s) = \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)\Gamma(d+is)}{\Gamma(2is)},$$

where $\operatorname{Re} a$, $\operatorname{Re} b$, $\operatorname{Re} c$, $\operatorname{Re} d > 0$ and all parameters are real, or a, b are real, $d = \overline{c}$, or $b = \overline{a}$, $d = \overline{c}$. Wilson polynomials are *even* polynomials given by

$$P_n(a,b,c,d;s^2) =$$
$$= (a+b)_n(a+c)_n(a+d)_n \, {}_4F_3\left[\begin{matrix} -n, n+a+b+c+d-1, a+is, a-is \\ a+b, a+c, a+d \end{matrix}; 1\right].$$

They satisfy to the difference equation

$$\mathcal{L}P_n = n(a+b+c+d-1)P_n.$$

1.3. Sturm–Liouville problems with continuous spectra. I know two solved problems.

a) We consider *even* functions $f(s)$ on the line, and

$$\mu(s) = \frac{\Gamma(a+is)\Gamma(b+is)}{\Gamma(2is)},$$

where a, $b > 0$. Let $\mathcal{L}$ be the same as above.

We consider the operator (it is called the inverse Olevsky transform, [26], or the inverse Jacobi transform, [14]):

$$L^2\left(\mathbb{R}_+, \frac{1}{\pi}\left|\frac{\Gamma(a+is)\Gamma(b+is)}{\Gamma(2is)}\right|^2 ds\right) \to L^2(\mathbb{R}_+, x^{a+b-1}(1+x)^{a-b})$$

defined by

$$Jf(x) = \frac{1}{\pi\Gamma(a+b)}\int_0^\infty {}_2F_1\left[\begin{matrix} a+is, a-is \\ a+b \end{matrix}; x\right] f(x) \left|\frac{\Gamma(a+is)\Gamma(b+is)}{\Gamma(2is)}\right|^2 ds.$$

The J send the difference operator $\mathcal{L}$ to the operator

$$Mf(x) = xf(x).$$

See [20], Theorem 2.1, but this is very special case of Cherednik, [4].

b) Let

$$\mu(s) = \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)}{\Gamma(d+is)\Gamma(2is)}.$$

In this case the spectral decomposition was done by an integral operator, whose kernel is a ${}_4F_3$-function, see Groenevelt [10], the discrete part of the spectrum was found in [21].

1.4. Partially solved problems. Romanovski-type systems of orthogonal polynomials. Romanovski [29] constructed orthogonal polynomials on $\mathbb{R}$ with respect to the weight $(1+ix)^{-a}(1-ix)^{-\overline{a}}$ on $\mathbb{R}$ and with respect the weight $x^{a-1}(1+x)^{-b}$ on $(0,\infty)$. Since the weights have polynomial decreasing, these orthogonal systems are finite. However, Romanovski polynomials correspond to discrete part of spectra of certain Sturm–Liouville problems (see [7], XIII.8, [14], [23]).

Lesky (see, e.g., [17], [18]) constructed numerous Romanovski type polynomial systems related to difference Sturm–Liouville problems, his list contains several difference problems in imaginary direction[2].

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[2]More generally, Lesky's papers indicate numerous unsolved but (certainly) solvable Sturm-Liouville problems.

1.5. Multidimensional analogs. See [4], [5].

1.6. Results of the paper. In Section 2, we show that the operators (1.1)–(1.5) are formally symmetric. Next, we found spectral decomposition for several operators $\mathcal{L}$. In Sections 3, 4 we consider

$$\mu(s) = \frac{1}{\Gamma(is)} \qquad \text{and} \qquad \mu(s) = \frac{\Gamma(a+is)}{\Gamma(2is)}$$

respectively. In both cases the spectrum is the half-line $\lambda > 0$. The spectral decomposition is given respectively by the inverse Kontorovich–Lebedev transform and the inverse Wimp transform with Whittaker kernel. Note that in a certain sense these problems (involving Bessel functions $_0F_1$ and Kummer functions $_1F_1$) are simpler than neo-classical polynomial problems (involving the Gauss function $_2F_1$ and higher hypergeometric functions $_3F_2(1)$, $_4F_3(1)$).

Next (Section 5), we consider $L^2(\mathbb{R})$ with respect to the measure

$$e^{\pi s}|\Gamma(\alpha/2 + is)|^2 \, ds \tag{1.9}$$

and the difference operator

$$\mathcal{L}f(s) = i(\alpha/2 + is)f(s - i) + 2\cosh\varphi \, sh(s) - i(\alpha/2 - is)f(s + i). \tag{1.10}$$

The form of this operator slightly differs from (1.1)–(1.5).

In Section 6 we discuss an example of a symmetric non self-adjoint operator and its essentially self-adjoint extensions.

In all cases essential self-adjointness is derived from the explicit spectral decomposition. It is an interesting question to find a priory proofs.

We also note that the problem (1.9)–(1.10) is an analytic continuation of the Meixner–Pollaszek problem (1.6)–(1.7). The objects of Section 6 also are "analytic continuations from integer points[3]" of the Meixner–Pollaszek polynomials.

2 Preliminaries

2.1. Imaginary shift in L^2. We say that a function is holomorphic in a closed strip $|\operatorname{Im} s| \leqslant \alpha$ if it is holomorphic in a larger strip $|\operatorname{Im} s| < \alpha + \delta$.

Lemma 2.1 *Let $H \subset L^2(\mathbb{R})$ be the subspace in $L^2(\mathbb{R})$ consisting of functions $f(s)$ admitting holomorphic continuation to the strip $|\operatorname{Im} s| \leqslant 1$ and satisfying the condition $|f(s)| = O(s^{-1/2-\varepsilon})$ in this strip. The operators*

$$T_+f(s) = f(s + i), \qquad T_-f(s) = f(s - i)$$

defined on H are symmetric in $L^2(\mathbb{R})$.

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[3]i.e., construction of analytic continuation involves the Carlson theorem, see, e.g., [1], Theorem 2.8.1

PROOF.

$$\int_{-\infty}^{\infty} f(s+i)\,\overline{g(\overline{s})}\,ds = \int_{i-\infty}^{i+\infty} f(t)\overline{g(\overline{t}+i)}\,dt = \int_{-\infty}^{\infty} f(t)\overline{g(\overline{t}+i)}\,dt.$$

2.2. Lemma on symmetry. Now let $\mu(s)$, $\nu(s)$ be the same as above, see (1.1)–(1.2). Therefore the weight $w(s)$ is

$$w(s) = \mu(s)\nu(s) = \frac{1}{2\pi} e^{2cs} \frac{\prod_{k=1}^{m} \Gamma(a_k + is)\Gamma(\overline{a}_k + -is)}{\prod_{l=1}^{n} \Gamma(b_l + is)\Gamma(\overline{b}_l - is)}.$$

For real s we can represent $w(s)$ in the form

$$w(s) = \frac{1}{2\pi} e^{2cs} \left| \frac{\prod_{k=1}^{m} \Gamma(a_k + is)}{\prod_{l=1}^{n} \Gamma(b_l + is)} \right|^2.$$

Let $A(s)$, $B(s)$ be as above

$$A(s) := \frac{\nu(s+i)}{\nu(s)} = e^{-ic} \frac{\prod_{k=1}^{m}(\overline{a}_k - is)}{\prod_{l=1}^{n}(\overline{b}_l - is)}$$

$$B(s) := \frac{\mu(s-i)}{\mu(s)} = e^{ic} \frac{\prod_{k=1}^{m}(a_k + is)}{\prod_{l=1}^{n}(b_l + is)}.$$

By (1.8), we have the following asymptotics of $w(s)$ in any strip $|\operatorname{Im} s| < \alpha$

$$w(s) \sim \Psi(s) := \text{const} \cdot |s|^{\sum(2\operatorname{Re} a_k - 1) - \sum(2\operatorname{Re} b_l - 1)} \exp\big(2cs + (n-m)\pi s\big), \quad (2.1)$$

as $s \to \infty$. We say that a function f is *w-decreasing* in a strip $|\operatorname{Im} s| \leqslant \alpha$ if

$$f(s) = O\left(\Psi(s)^{-1/2} s^{-m-1/2-\varepsilon}\right), \qquad s \to \infty.$$

This condition provides

$$f(s+i\beta),\ A(s)f(s+i\beta),\ B(s)f(s+i\beta) \in L^2(\mathbb{R}, w(s)\,ds)$$

for all β satisfying $|\beta| \leqslant \alpha$. Denote by $\mathcal{H}[w]$ the space of all functions holomorphic in the strip $|\operatorname{Im} s| \leqslant 1$ and w-decreasing in this strip. $\qquad\square$

Lemma 2.2 *Let*

$$\operatorname{Re} a_j > 0$$

for all j. The operator

$$Rf(s) = A(s)f(s+i)$$

defined on the domain $\mathcal{H}[w]$ is symmetric in $L^2(\mathbb{R}, w(s)\,ds)$.

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PROOF. We verify the identity $\langle Rf, g \rangle = \langle f, Rg \rangle$ for $f, g \in \mathcal{H}[w]$:

$$\int_{-\infty}^{\infty} \frac{\nu(s+i)}{\nu(s)} f(s+i) \overline{g(\overline{s})} \mu(s)\nu(s)\, ds = \int_{-\infty}^{\infty} f(s+i) \overline{g(\overline{s})} \mu(s)\nu(s+i)\, ds =$$

$$= \int_{i-\infty}^{i+\infty} f(s) \overline{g(\overline{s}+i)} \mu(s-i)\nu(s)\, ds = \int_{-\infty}^{\infty} f(s) \overline{g(\overline{s}+i)} \mu(s-i)\nu(s)\, ds =$$

$$= \int_{-\infty}^{\infty} f(s) \frac{\mu(s-i)}{\mu(s)} \overline{g(\overline{s}+i)} \mu(s)\nu(s)\, ds = \int_{-\infty}^{\infty} f(s) \frac{\overline{\nu(\overline{s}+i)}}{\nu(\overline{s})} \overline{g(\overline{s}+i)} \mu(s)\nu(s)\, ds.$$

The condition $\operatorname{Re} a_j > 0$ provides absence of poles of $\nu(s+i)\mu(s)$ in the strip $0 < \operatorname{Im} s < 1$.

Corollary 2.3 *Under the same conditions the operator*

$$\mathcal{L}f(s) = A(s)f(s+i) - (A(s)+B(s))f(s) + B(s)f(s-i)$$

is symmetric on the subspace $\mathcal{H}[w] \subset L^2(\mathbb{R}, w(s)\, ds)$.

2.3. Change of a weight. Let $w_2(s) = \tau(s)\overline{\tau(\overline{s})})w_1(s)$. Then the operator

$$Hf(s) = \tau(s)f(s)$$

is a unitary operator $L^2(\mathbb{R}, w_2(s)) \to L^2(\mathbb{R}, w_1(s))$. Evidently,

$$H^{-1}T_+Hf(s) = \frac{\tau(s+i)}{\tau(s)}T_+, \qquad H^{-1}T_-Hf(s) = \frac{\tau(s-i)}{\tau(s)}T_+.$$

2.4. Operators in $L^2(\mathbb{R})$.

Lemma 2.4 *Let an operator*

$$Rf(s) = L(s)f(s+i)$$

be formally symmetric in $L^2(\mathbb{R}, ds)$. *Then*

$$L(s) = \overline{L(\overline{s}-i)}. \tag{2.2}$$

This is straightforward.

Note that if $L(s)$ satisfy (2.2), then $L(s)^{-1}$ satisfy the same condition. Also, if $L_1(s)$, $L_2(s)$ satisfy (2.2), then $L_1(s)L_2(s)$ satisfy (2.2).

Obvious solutions are

$$L(s) = i/2 + s,$$
$$L(s) = (i/2 + ia + s)(i/2 - ia + s),$$
$$L(s) = h(e^{2\pi s}).$$

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3 The Kontorovich–Lebedev transform

3.1. Difference operator. Now $\mu(s) = \Gamma(is)$, $w(s) = |\Gamma(is)|^{-2}$. We consider the space of *even* functions, $f(s) = f(-s)$, the inner product is given by

$$\langle f, g \rangle = \frac{2}{\pi} \int_{-\infty}^{\infty} f(s) \overline{g(\overline{s})} \frac{ds}{|\Gamma(is)|^2} = \frac{2}{\pi^2} \int_{-\infty}^{\infty} f(s) \overline{g(\overline{s})} s \sinh(\pi s) \, ds.$$

We consider a difference operator $\mathcal{L}$ given by

$$\mathcal{L}f(s) = \frac{1}{is}\big(f(s+i) - f(s-i)\big) \tag{3.1}$$

defined on the subspace $\mathcal{H}[w] \subset L^2(\mathbb{R}_+, |\Gamma(is)|^{-2}ds)$

Lemma 3.1 *The operator $\mathcal{L}$ is essentially self-adjoint.*

The spectral decomposition is given by the inverse Kontorovich–Lebedev transform, see the next subsection.

3.2. The Kontorovich–Lebedev transform. Preliminaries. The *Macdonald functions* $K_\nu(z)$ are solutions of the modified Bessel differential equation (see [9],7.2(11)), i.e. the equation

$$\left(z\frac{d}{dz}\right)^2 g(z) - z^2 g(z) = -\nu^2 g(z).$$

They are defined by (see [9], 7.2(13)),

$$K_\nu(z) = \frac{\pi}{\sin(\nu\pi)}(I_{-\nu}(z) - I_\nu(z)),$$

where $I_\nu(z)$ are the modified Bessel functions,

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(ze^{i\pi/2}) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}.$$

For each $z \neq 0$ the function $h_z(\nu) := K_\nu(z)$ is an entire function of the variable ν,

$$K_\nu(z) = K_{-\nu}(z).$$

For positive $z \in \mathbb{R}$ and $\nu \in i\mathbb{R}$ values of $K_\nu(z)$ are real.

Below we use two identities (see [34], (3.71.1)–(3.71.2))

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_\nu(z), \tag{3.2}$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -\frac{d}{dz}K_\nu(z). \tag{3.3}$$

The Kontorovich–Lebedev transform [15], [16], Section 6.5, [37] is given by[4] the formula

$$\mathfrak{K}g(s) = \int_0^\infty K_{is}(x)g(x)\frac{dx}{x}. \tag{3.4}$$

The inverse transform is

$$\mathfrak{K}^{-1}f(x) = \frac{2}{\pi}\int_0^\infty f(s)K_{is}(x)\frac{ds}{|\Gamma(is)|^2}. \tag{3.5}$$

The Kontorovich–Lebedev transform is a unitary operator

$$L^2(\mathbb{R}_+, x^{-1}dx) \to L^2(\mathbb{R}_+, 2\pi^{-1}|\Gamma(is)|^{-2}ds).$$

3.3. The statement.

Theorem 3.2 *The Kontorovich-Lebedev transform provides a unitary equivalence between the operator*

$$Pg(x) = \frac{2}{x}g(x)$$

in $L^2(\mathbb{R}_+, x^{-1}dx)$ and the operator $\mathcal{L}$ given by (3.1).

PROOF. We use (3.2),

$$\int_0^\infty \frac{2}{x}g(x) \cdot K_{is}(x)\frac{dx}{x} = \int_0^\infty g(x) \cdot \frac{2K_{is}(x)}{x}\frac{dx}{x} =$$

$$= \int_0^\infty g(x)\frac{1}{is}\big(K_{i(s+i)}(x) - K_{i(s-i)}(x)\big)\frac{dx}{x} = \frac{1}{is}\big(\mathfrak{K}g(s+i) - \mathfrak{K}g(s-i)\big).$$

This proves the statement.

REMARK. However, Lemma 3.1 in this moment is not proved, it a special case of Lemma 4.1 proved below.

3.4. An additional remark. Applying (3.3), we get the following statement

Proposition 3.3 *The Kontorovich–Lebedev transform send the operator*

$$Qg(x) = \left(\frac{d}{dx} - \frac{1}{x}\right)g(x)$$

to the operator

$$\mathcal{M}f(s) = \frac{1}{2}\big(f(s+i) - f(s-i)\big).$$

Therefore, we can evaluate the image of any operator $x^{-m}\frac{d^n}{dx^n}$ under the Kontorovich–Lebedev transform.

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[4]Here and below we understand integral operators in the sense of the kernel theorem, see, e.g., [11], Section 5.2. However, for the Kontorovich–Lebedev transform and the Wimp transform discussed below conditions of literal validness of formulas are well investigated.

4 The Wimp transform

4.1. Difference problem. Now $\mu(s) = \frac{\Gamma(1/2-\rho+is)}{\Gamma(2is)}$, we consider the space of *even* functions on $\mathbb{R}$ with inner product

$$\langle f, g \rangle = \frac{1}{4\pi} \int_{-\infty}^{\infty} f(s)\overline{g(\overline{s})} \left| \frac{\Gamma(1/2 - \rho + is)}{\Gamma(2is)} \right|^2 ds.$$

We consider the following difference operator

$$\mathcal{L}f(s) = \frac{1 - \rho - is}{(-2is)(1 - 2is)} f(s + i) - $$
$$- \left(\frac{1 - \rho - is}{(-2is)(1 - 2is)} + \frac{1 - \rho + is}{(+2is)(1 + 2is)} \right) f(s) + \frac{1 - \rho + is}{(+2is)(1 + 2is)} f(s - i). \tag{4.1}$$

As above, this operator is defined on the subspace $\mathcal{H}[w] \subset L^2(\mathbb{R}, w(s)ds)$.

Lemma 4.1 *Let $\rho < 1/2$. Then the operator $\mathcal{L}$ is essentially self-adjoint.*

4.2. Whittaker functions and the Wimp transform. Preliminaries.
Whittaker functions $W_{\rho,\sigma}(z)$ are versions of confluent hypergeometric functions. They are solutions of the Whittaker equation (see [8], 6.1 (4))

$$\left(x^2 \frac{d^2}{dx^2} - \frac{x^2}{4} + \rho x \right) f(x) = (\sigma^2 - 1/4)f(x). \tag{4.2}$$

The explicit expression is

$$W_{\rho,\sigma}(x) = e^{-x/2} \Big(\frac{\Gamma(-2\sigma)x^{1/2+\sigma}}{\Gamma(1/2 - \rho - \sigma)} \, {}_1F_1 \begin{bmatrix} 1/2 - \rho + \sigma \\ 1 + 2\sigma \end{bmatrix}; x \Big] + $$
$$+ \frac{\Gamma(2\sigma)x^{1/2-\sigma}}{\Gamma(1/2 - \rho + \sigma)} \, {}_1F_1 \begin{bmatrix} 1/2 - \rho - \sigma \\ 1 - 2\sigma \end{bmatrix}; x \Big] \Big), \tag{4.3}$$

see [30], (1.9.10). There are the following integral representations (see [8], 6.11(18), [27], 2.3.6.9),

$$W_{\rho,\sigma}(x) = \frac{e^{-x/2}x^{\rho}}{\Gamma(1/2 - \rho + \sigma)} \int_0^{\infty} e^{-xt} t^{-1/2-\rho+\sigma}(1 + t)^{-1/2+\rho+\sigma} dt. \tag{4.4}$$

and the Barnes representation (see [28], 8.4.44.3, [30],(3.5.16)),

$$W_{\rho,\sigma}(z) = \frac{e^{-x/2}}{2\pi\Gamma(1/2 - \rho - \sigma)\Gamma(1/2 - \rho + \sigma)} \times $$
$$\times \int_{-\infty}^{\infty} \Gamma(it + 1/2 + \sigma)\Gamma(it + 1/2 - \sigma)\Gamma(-\rho - it)x^{-it} dt. \tag{4.5}$$

REMARK. If $\rho \in \mathbb{R}$, $\sigma \in i\mathbb{R}$, $x > 0$, then $W_{\rho,\sigma}(x)$ is real. This follows from (4.3). $\square$

Fix real $\rho < 1/2$. The *Wimp transform* $\mathfrak{W}_\rho$ is the integral operator given by

$$\mathfrak{W}_\rho g(s) = \int_0^\infty g(x) W_{\rho,is}(x) \frac{dx}{x^2}$$

(see [36], [37]). The inverse transform is

$$\mathfrak{W}_\rho^{-1} f(x) = \frac{1}{2\pi} \int_0^\infty f(x) W_{\rho,is}(x) \left| \frac{\Gamma(1/2 - \rho + is)}{\Gamma(2is)} \right|^2 ds.$$

The Wimp transform is a unitary operator

$$L^2(\mathbb{R}_+, x^{-2}dx) \to L^2\left(\mathbb{R}_+, \frac{1}{2\pi} \left| \frac{\Gamma(1/2 - \rho + is)}{\Gamma(2is)} \right|^2 ds\right).$$

REMARK. This theorem can be obtained by writing of explicit spectral decomposition of the differential operator (4.2) as it is explained in [7], Chapter XIII. $\square$

The Macdonald function K_ν admits the following expression in the terms of Whittaker functions:

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} W_{0,\nu}(x).$$

Therefore the Kontorovich–Lebedev transform is a special case of Wimp transforms.

4.3. The statement.

Theorem 4.2 *The Wimp transform send the operator*

$$Rg(x) = x^{-1}g(x) \tag{4.6}$$

to the difference operator $\mathcal{L}$ defined by (4.1).

Theorem is a corollary of the following lemma.

Lemma 4.3 *The Whittaker functions satisfy the difference equation*

$$\frac{1-\rho-\sigma}{(-2\sigma)(1-2\sigma)}(W_{\rho,\sigma-1}(x) - W_{\rho,\sigma}) + \frac{1-\rho+\sigma}{(2\sigma)(1+2\sigma)}(W_{\rho,\sigma+1}(x) - W_{\rho,\sigma}) =$$

$$= \frac{1}{x} W_{\rho,\sigma}(x). \tag{4.7}$$

PROOF. We use the Barnes integral (4.5). We multiply both sides of (4.7) by $e^{x/2}$ and pass to Mellin transforms (see below (5.2)–(5.3)). Denote by $h(t)$ the Mellin transform of $e^{x/2}W_{\rho,\sigma}(x)$, i.e.,

$$h(t) = \frac{\Gamma(it + 1/2 + \sigma)\Gamma(it + 1/2 - \sigma)\Gamma(-\rho - it)}{\Gamma(1/2 - \rho - \sigma)\Gamma(1/2 - \rho + \sigma)}.$$

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The Mellin transforms of $e^{x/2}W_{\rho,\sigma\pm1}(x)$ are $\gamma_{\pm}(t)h(t)$, where

$$\gamma_{\pm}(t) = \frac{(-1/2 - \rho \mp \sigma)(t + 1/2 \pm \sigma)}{(1/2 - \rho \pm \sigma)(t - 1/2 \mp \sigma)}.$$

In the left-hand side we get

$$h(s) \cdot \left\{ \frac{1 - \rho - \sigma}{(-2\sigma)(1 - 2\sigma)}(\gamma_{-}(t) - 1) + \frac{1 - \rho + \sigma}{(2\sigma)(1 + 2\sigma)}(\gamma_{+}(t) - 1) \right\} =$$

$$= h(s)\frac{-t - \rho}{(t - 1/2 - \sigma)(t - 1/2 + \sigma)} =$$

$$= \frac{\Gamma(it - 1/2 + \sigma)\Gamma(it - 1/2 - \sigma)\Gamma(-\rho - it + 1)}{\Gamma(1/2 - \rho - \sigma)\Gamma(1/2 - \rho + \sigma)} = h(s + i).$$

Shift of a Mellin transform by i is equivalent to multiplication of the original by $1/x$. $\qquad\square$

4.4. Proof of self-adjointness. The space $C_c^{\infty}(\mathbb{R}_+)$ of smooth functions with compact support on $(0, \infty)$ is a domain of essential self-adjointness of the operator (4.6). It is sufficient to prove the following lemma.

Lemma 4.4 $\mathfrak{W}_{\rho}\big(C_c^{\infty}(\mathbb{R}_+)\big) \subset \mathcal{H}[w]$.

Lemma 4.5 *Fix $\rho < 1/2$. For (σ, x) ranging in a domain*

$$|\operatorname{Re}\sigma| \leqslant 1, \qquad 0 < c \leqslant x \leqslant C < \infty \tag{4.8}$$

the following uniform estimate holds

$$|W_{\rho,\sigma}(x)| = O\left(e^{\pi|\operatorname{Im}\sigma|/2}|\operatorname{Im}\sigma|^{\rho+1}\right). \tag{4.9}$$

PROOF OF LEMMA 4.5. The integral formula (4.4) converges if $\operatorname{Re}\sigma > \rho - 1/2$ and admits the holomorphic to the whole plane $\sigma \in \mathbb{C}$.

The statement is very simple if $\rho < -1/2$ (the integral in (4.4) is bounded and the desired estimate is obtained from an estimate of a pre-integral factor. But we wish to cover also the interval $-1/2 < \rho < 1/2$.

Fix $A > B > 1$. Represent 1 as $1 = \varphi(t) + \psi(t)$, where φ, $\psi(t) \geqslant 0$ are smooth nonnegative on $\mathbb{R}_+$, $\psi(t) = 0$ for $t < A$, and $\varphi = 0$ for $t > B$. We write the integral in (4.4) as

$$\int_0^{\infty} = \int_0^A e^{-xt}t^{-1/2-\rho+\sigma}(1 + t)^{-1/2+\rho+\sigma}\varphi(t)\, dt+$$

$$+ \int_B^{\infty} e^{-xt}t^{-1/2-\rho+\sigma}(1 + t)^{-1/2+\rho+\sigma}\psi(t)\, dt. \tag{4.10}$$

The second summand is uniformly bounded in our domain (4.8), the integrand is dominated by

$$e^{-ct}t^{1/2-\rho}(1 + t)^{1/2+\rho}$$

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Next, we represent the first summand of (4.10) as

$$\int_0^A = \int_0^A t^{-1/2-\rho+\sigma}\left(e^{-xt}(1+t)^{-1/2+\rho+\sigma}\varphi(t) - 1\right)dt + \int_0^A t^{-1/2-\rho+\sigma}\,dt$$

Denote by $Q(t,x,\sigma)$ the first integrand. Then $|Q(t,x,\sigma)|$ depend on t, x, $\mathrm{Re}\,\sigma$, these variables range in a compact set, the function Q is continuous on this set. Therefore first summand is uniformly bounded in (4.8), the second summand is uniformly bounded in (4.8) outside a neighborhood of $\sigma = \rho - 1/2$.

Thus $\int_0^\infty$ is uniformly bounded in in (4.8) outside a neighborhood of $\sigma = \rho - 1/2$.

Next, we multiply the integral (4.4) by the pre-integral factor $\frac{e^{-x/2}x^\rho}{\Gamma(1/2-\rho+\sigma)}$. Since $\mathrm{Re}(1/2-\rho+\sigma) \in (-1/2-\rho, 3/2-\rho)$, we have

$$\Gamma(1/2-\rho+\sigma)^{-1} = O(e^{\pi|\mathrm{Im}\,\sigma|/2}|\mathrm{Im}\,\sigma|^{\rho+1}), \qquad |\mathrm{Im}\,\sigma| \to \infty$$

and we get (4.9). $\qquad\qquad\square$

PROOF OF LEMMA 4.4. By Lemma 4.5, for a function $f \in C_c^\infty(\mathbb{R}_+)$ with compact support, we have

$$|\mathfrak{W}_\rho f(s)| \leqslant C \cdot e^{\pi|\mathrm{Re}\,s|/2}|\mathrm{Re}\,s|^{\rho+1/2}. \qquad (4.11)$$

Next, we use (4.2),

$$-(1/4+s^2)\mathfrak{W}_\rho f(s) = \int_0^\infty (-1/4 - s^2)W_{\rho,is}(x)) \cdot f(x)\frac{dx}{x} =$$

$$= \int_0^\infty \left(x^2\frac{d^2}{dx^2} - \frac{x^2}{4} + \rho x\right)W_{\rho,is}(x) \cdot f(x)\frac{dx}{x} =$$

$$= \int_0^\infty W_{\rho,is}(x)\cdot\left[x\frac{d^2}{dx^2}(xf(x)) - \frac{1}{4}x^2 f(x) + \rho x f(x)\right]\frac{dx}{x}.$$

We apply (4.11) for the function in square brackets and get

$$|\mathfrak{W}_\rho f(s)| \leqslant C \cdot (s^2 + 1/4)^{-1} \cdot e^{\pi|\mathrm{Re}\,s|/2}|\mathrm{Re}\,s|^{\rho+1}$$

and $\mathfrak{W}_\rho f(s) \in \mathcal{H}[w]$. $\qquad\qquad\square$

5　The Vilenkin transform

5.1. Difference problem. Fix $\alpha > 0$, $\varphi > 0$. We consider the weight

$$w(t) = \frac{1}{2\pi}|\Gamma(\alpha/2 + it)|^2 e^{\pi t},$$

the corresponding space $L^2(\mathbb{R}, w(t)\,dt)$, and the difference operator

$$\mathcal{L}f(t) = i(\alpha/2 + it)f(t-i) + 2\cosh\varphi\,th(t) - i(\alpha/2 - it)f(t+i).$$

This operator differs from (1.1)-(1.5), but it is symmetric (proof is the same as in Lemma 2.2).

Theorem 5.1 *The operator $\mathcal{L}$ is essentially self-adjoint on the space $\mathcal{H}[w]$.*

5.2. The Vilenkin transform.

Theorem 5.2 *The Vilenkin transform*

$$\mathfrak{V}_\alpha g(t) = (1 - e^{-2\varphi})^{\alpha/2} e^{-\varphi i t} \int\limits_{-\infty}^{\infty} g(s) \,_2F_1 \left[\begin{matrix} \alpha/2 - is, \alpha/2 + it \\ \alpha \end{matrix} ; 1 - e^{-2\varphi} \right] w(s) \, ds.$$

is a unitary operator $L^2(\mathbb{R}, w(s)\,ds) \to L^2(\mathbb{R}, w(s)\,ds)$.

This is a minor modification of Vilenkin [32], §7.4, see also [33], 7.7.7.
Since the operator $\mathfrak{V}$ is unitary, the inversion formula is

$$\mathfrak{V}_\alpha^{-1} f(s) = (1 - e^{-2\varphi})^{\alpha/2} \int\limits_{-\infty}^{\infty} f(t) \,_2F_1 \left[\begin{matrix} \alpha/2 + is, \alpha/2 - it \\ \alpha \end{matrix} ; 1 - e^{-2\varphi} \right] e^{\varphi i t} w(t) \, dt.$$

Theorem 5.3 *The inverse Vilenkin transform $\mathfrak{V}_\alpha^{-1}$ send the operator $\mathcal{L}$ to the operator*

$$\mathcal{N} f(s) = 2s \sinh \varphi f(s).$$

To prove these statements, we decompose the Vilenkin transform as a product of 3 simple transformations, see below formula (5.11).

5.3. Highest weight representations of $\mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}(2, \mathbb{R})$ is the group of 2×2 real matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det g = 1$. Denote by Π the half-plane $\operatorname{Im} z > 0$.

Fix $\alpha > 0$. Consider the Hilbert space H_α of holomorphic functions on Π determined by reproducing kernel (see, e.g., [24], Section 7.1)

$$K(z, u) = \left(\frac{z - \overline{u}}{2i} \right)^{-\alpha}.$$

In other words, denote $\Psi_a(z) := K(z, a)$. Then for any $F \in H_\alpha$ we have

$$\langle F, \Psi_a \rangle = F(a). \tag{5.1}$$

For $\alpha > 1$ the inner product in H_α admits the following integral representation

$$\langle F, G \rangle = \mathrm{const}(\alpha) \int_\Pi F(z) \overline{G(z)} (\operatorname{Im} z)^{\alpha - 2} dz \, d\overline{z}.$$

Consider the following operators in H_α

$$T_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(z) = F \left(\frac{b + zd}{a + zc} \right) (a + zc)^{-\alpha}.$$

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The function $(a + zc)^{-\alpha}$ is multi-valued. We choose arbitrary branch of this function on Π. Then operators $T_\alpha(g)$ are unitary and satisfy the condition

$$T_\alpha(g_1)T_\alpha(g_2) = \lambda(g_1, g_2)T_\alpha(g_1 g_2),$$

where $\lambda(g_1, g_2) \in \mathbb{C}$. Thus we get a projective unitary representation of $\mathrm{SL}_2(\mathbb{R})$, such representations are called *highest weight representations*.

5.4. The Mellin transform. Preliminaries. See, e.g., [31]. For a function f on $\mathbb{R}_+$ we define a Mellin transform $\mathfrak{M}f(s)$ as

$$\mathfrak{M}f(s) = \int_0^\infty f(x)x^{is-1}dx. \tag{5.2}$$

The inverse transform is given by

$$\mathfrak{M}^{-1}g(x) = \frac{1}{2\pi} \int_{-\infty}^\infty g(s)x^{-is}ds. \tag{5.3}$$

The Mellin transform is a unitary operator $L^2(\mathbb{R}_+, x^{-1}dx) \to L^2(\mathbb{R}, \frac{1}{2\pi}ds)$.

Notice, that changing variable $x = e^t$ in (5.2), we come to the usual Fourier transform.

5.5. Spectral decomposition of dilatation operators. Consider a one-parametric subgroup $A \simeq \mathbb{R}_+^\times$ in $\mathrm{SL}_2(\mathbb{R})$ consisting of matrices of the form $D(a) = \begin{pmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{pmatrix}$, where $a > 0$. The subgroup A acts in the space H_α by the transformations

$$T_\alpha(D(a)) = f(z) = f(a^{-1}z)a^{-\alpha/2}. \tag{5.4}$$

Next, consider the measure $d\mu(s)$ on $\mathbb{R}$ given by

$$\mu(s)\,ds := \frac{1}{2\pi\Gamma(\alpha)}|\Gamma(\alpha/2 + is)|^2\,ds$$

and the action of the same group in the space $L^2(\mathbb{R}, \mu(s)\,ds)$ given by the formula

$$\tau_\alpha(D(a))f(s) = f(s)a^{is}. \tag{5.5}$$

Consider the operator $J : L^2(\mathbb{R}, \mu(s)\,ds) \to H_\alpha$ given by

$$F(z) = J_\alpha f(z) = \frac{2^\alpha}{2\pi\Gamma(\alpha)} \int_{-\infty}^\infty f(s)\left(\frac{z}{i}\right)^{-\alpha/2-is} |\Gamma(\alpha/2 + is)|^2 ds, \tag{5.6}$$

we choose a branch of $(z/i)^{-\alpha/2-is} = e^{-(\alpha/2+is)\ln(z/i)}$ such that $\ln z/i$ is real for $z = ip$, $p > 0$.

Therefore $F(ip)(p)^{\alpha/2}$ is the inverse Mellin transform of $\frac{2^\alpha}{\Gamma(\alpha)}f(s)|\Gamma(\alpha/2 + is)|^2$.

Applying the direct Mellin transform, we get

$$f(s) \cdot \frac{2^\alpha}{\Gamma(\alpha)}|\Gamma(\alpha/2 + is)|^2 = \int_0^\infty F(ip)\,p^{\alpha/2+is-1}dp. \tag{5.7}$$

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Proposition 5.4 *The transform J_α is a unitary operator*

$$J_\alpha : L^2(\mathbb{R}, \mu(s)\, ds) \to H_\alpha$$

intertwining actions (5.4) *and* (5.5).

PROOF. A verification of

$$J \circ \tau_\alpha(D(a)) = T_\alpha(D(a)) \circ J$$

is straightforward. Next,

$$T_\alpha(D(a))\Psi_i = a^{\alpha/2}\Psi_{ai}.$$

By (5.1), the system of vectors Ψ_{ai}, where $a > 0$ is total in the Hilbert space H_α.

Next, we consider functions

$$\Phi_a := a^{\alpha/2 + is}$$

in $L^2(\mathbb{R}, \mu(s)\, ds)$. Then

$$\tau_\alpha(D(a))\Phi_1 = \Phi_a \cdot a^{\alpha/2}.$$

To prove unitarity, it is sufficient to show (see, e.g., [24], Theorem 7.1.4) that

$$J\Phi_a = \Psi_{ia}, \tag{5.8}$$

$$\langle \Phi_a, \Phi_b \rangle_{L^2} = \langle \Psi_{ia}, \Psi_{ib} \rangle_{V_\alpha} = \left(\frac{a+b}{2}\right)^{-\alpha}. \tag{5.9}$$

First, note that

$$\int_0^\infty (1+x)^{-\alpha} x^{-is}\, dx = \mathrm{B}(is, \alpha - is) = \frac{\Gamma(is)\Gamma(\alpha - is)}{\Gamma(\alpha)}.$$

Applying the inversion formula for the Mellin transform, we get ([28], 8.5.2.5)

$$\frac{1}{2\pi}\int_{-\infty}^\infty \Gamma(is)\Gamma(\alpha - is) x^{is-1}\, ds = \Gamma(\alpha)(1+x)^{-\alpha}.$$

Both formulas (5.8)–(5.9) are reduced to the latter integral. $\qquad\square$

5.6. Calculation. Proof of Theorem 5.2 Set

$$r_\varphi := \frac{1}{\sqrt{2\sinh\varphi}}\begin{pmatrix} 1 & 1 \\ e^{-\varphi} & e^\varphi \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{5.10}$$

Lemma 5.5 *The operator*

$$J_\alpha^{-1} T_\alpha(r_\varphi) J_\alpha f(t) = (2\sinh\varphi)^{\alpha/2} e^{-\varphi(\alpha/2 + it)} e^{\pi t/2} \times$$

$$\times \int_{-\infty}^\infty f(s)\, {}_2F_1\left[\begin{matrix} \alpha/2 - is, \alpha/2 + it \\ \alpha \end{matrix}; 1 - e^{-2\varphi}\right] e^{-\pi s/2} \mu(s)\, ds. \tag{5.11}$$

is a unitary operator $L^2(\mathbb{R}, \mu(s)\, ds) \to L^2(\mathbb{R}, \mu(s)\, ds)$.

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PROOF. The operator
$$J_\alpha^{-1} T_\alpha(r_\varphi) J_\alpha$$
is unitary by definition as a product of 3 unitary operators
$$L^2(\mathbb{R}, \mu(s)\, ds) \to H_\alpha \to H_\alpha \to L^2(\mathbb{R}, \mu(s)\, ds).$$

We must find explicit formula for composition. Write J_α in the form

$$J_\alpha f(z) = e^{i\pi\alpha/4} 2^\alpha \int_{-\infty}^{\infty} f(s) z^{-\alpha/2 - is} e^{-\pi s/2} d\mu(s), \qquad (5.12)$$

we use $(e^{-i\pi/2})^{-is} = e^{-\pi s/2}$. In this formula we take the branch of $z^{-\alpha/2 - is}$ given by

$$z^{-\alpha/2 - is} = e^{-(\alpha/2 + is)\ln z}, \qquad (5.13)$$

where the logarithm is real on the semi-axis $z > 0$. Then the inversion formula is

$$J_\alpha^{-1} F(t) = \frac{e^{-i\pi\alpha/4} 2^{-\alpha} \Gamma(\alpha) e^{\pi t/2}}{\Gamma(\alpha/2 + it)\Gamma(\alpha/2 - it)} \int_0^\infty F(z) z^{\alpha/2 + it - 1} ds. \qquad (5.14)$$

Recall that r_φ is given by (5.10),

$$T_\alpha(r_\varphi) J_\alpha f(z) = e^{i\pi\alpha/4} 2^\alpha (2\sinh\varphi)^{\alpha/2} \times$$

$$\times \int_{-\infty}^{\infty} \left(\frac{e^\varphi z + 1}{e^{-\varphi} z + 1} \right)^{\alpha/2 - is} (e^{-\varphi} z + 1)^{-\alpha} f(s) e^{-\pi s/2} \, d\mu(s) =$$

$$= \frac{e^{i\pi\alpha/4} 2^\alpha (2\sinh\varphi)^{\alpha/2}}{2\pi\Gamma(\alpha)} \times$$

$$\times \int_{-\infty}^{\infty} (e^\varphi z + 1)^{-\alpha/2 - is} (e^{-\varphi} z + 1)^{-\alpha/2 + is} f(s) \, e^{-\pi s/2} |\Gamma(\alpha + is)|^2 ds. \quad (5.15)$$

Next, we apply the inverse transform J_α^{-1},

$$J_\alpha^{-1} T_\alpha(r_\varphi) J_\alpha f(t) = \frac{(2\sinh\varphi)^{\alpha/2} e^{\pi t/2}}{\Gamma(\alpha/2 + it)\Gamma(\alpha/2 - it)} \times$$

$$\times \int_0^\infty \int_{-\infty}^{\infty} z^{\alpha/2 + it - 1} (e^\varphi z + 1)^{-\alpha/2 - is} (e^{-\varphi} z + 1)^{-\alpha/2 + is} f(s) \times$$

$$\times e^{-\pi s/2} |\Gamma(\alpha + is)|^2 ds\, dz. \quad (5.16)$$

We must evaluate the integral in z,

$$\int_0^\infty z^{\alpha/2 + it - 1} (e^\varphi z + 1)^{-\alpha/2 - is} (e^{-\varphi} z + 1)^{-\alpha/2 + is} \, dz =$$

$$= e^{-\varphi(\alpha/2 + it)} \int_0^\infty u^{\alpha/2 + it - 1} (1 + u)^{-\alpha/2 - is} (1 + e^{-2\varphi} u)^{-\alpha/2 + is} \, du =$$

$$= e^{-\varphi(\alpha/2 + it)} \cdot \frac{\Gamma(\alpha/2 + it)\Gamma(\alpha/2 - it)}{\Gamma(\alpha)} \, {}_2F_1 \left[\begin{matrix} \alpha/2 - is, \alpha/2 + it \\ \alpha \end{matrix} ; 1 - e^{-2\varphi} \right],$$

here we applied an integral representation of the Gauss hypergeometric function,

$$_2F_1\begin{bmatrix} a, b \\ c \end{bmatrix}; 1 - u \end{bmatrix} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty y^{b-1}(1+y)^{a-c}(1+yu)^{-a}dy, \qquad (5.17)$$

see [8], 2.12(5); this is valid for $|\arg u| < \pi$.

Thus, we get (5.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROOF OF THEOREM 5.2. Finally, we change function by the rule

$$g(s) = e^{-\pi s/2}f(s). \qquad (5.18)$$

This is equivalent to passing to the space $L^2(\mathbb{R}, w(s)\,ds)$, where $w(s) = e^{\pi s}d\mu(s)$.

5.7. Calculations. The difference operator. Now we evaluate the image of the operator

$$f(s) \mapsto sf(s)$$

under $J_\alpha^{-1}T_\alpha(r_\varphi)J_\alpha$. Differentiating (5.15) by parameter z, we get that $J_\alpha^{-1}T_\alpha(r_\varphi)$ send the operator

$$f(s) \mapsto -2is\sinh\varphi f(s)$$

to

$$D := (z^2 + 2z\cosh\varphi + 1)\frac{d}{dz} + \alpha(z + \cosh\varphi).$$

Next, we evaluate the corresponding operator in $L^2(\mathbb{R}, w(s)\,ds)$. First, set

$$g(t) = \int_0^\infty F(z)z^{\alpha/2+it-1}dz, \qquad h(t) = \frac{g(t)}{\Gamma(\alpha/2+is)\Gamma(\alpha/2-is)} \qquad (5.19)$$

and evaluate

$$\int_0^\infty DF(z)z^{\alpha/2+it-1}dz =$$

$$= \int_0^\infty F'(z)\left(z^{\alpha/2+it+1} + 2\cosh\varphi\, z^{\alpha/2+it} + z^{\alpha/2+it-1}\right)dz +$$

$$+ \alpha \int_0^\infty F(z)\left(z^{\alpha/2+it} + \cosh z^{\alpha/2+it-1}\right)dz. \qquad (5.20)$$

Next we *formally* integrate by parts and come to

$$(\alpha/2 - it - 1)g(t-i) - 2\cosh\varphi\, tg(t) + (-\alpha/2 - it - 1)g(t+i).$$

For functions $h \in L^2(\mathbb{R}, w(t)\,dt)$ we get the transformation

$$h(t) \mapsto (\alpha/2 + it)h(t-i) - 2i\cosh\varphi\, th(t) - (\alpha/2 - it)h(t+i).$$

5.8. Self-adjointness. Proof of Theorem 5.1. Denote by W_R the space of functions $f(s)$ holomorphic in the strip

$$|\operatorname{Im} s| < R \qquad (5.21)$$

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satisfying the condition: for any $A > 0$ there is C such that

$$|f(s)| < C \cdot \exp(-A|\operatorname{Re} s|).$$

The operator $f \mapsto sf(s)$ in $L^2(\mathbb{R}, d\mu(s))$ is essentially self-adjoint on W_R.

Theorem 5.1 is a corollary of the following lemma.

Lemma 5.6 *If R is sufficiently large, then for any $f \in W_R$ we have $\mathfrak{V}_\alpha f \in \mathcal{H}[w]$.*

PROOF. Since $f(z)$ super-exponentially decreases, $J_\alpha f(z)$, see (5.12)–(5.13), is a well-defined analytic function on the universal covering of $\mathbb{C} \setminus 0$. In other words, we can assume in (5.13) that $-\infty < \arg z < +\infty$. Since f is analytic in the strip, the Fourier transform of f exponentially decreases, therefore the Mellin transform decreases as $O(|z|^R)$ as $|z| \to 0$ and as $O(|z|^{-R})$ as $z \to \infty$ (see [31], Theorem 31), both $O(\cdot)$ are uniform in any sector $|\arg z| < C$ with finite central angle.

After the transform $T_\alpha(r_\varphi)$ we get a function $F(z) := T_\alpha(r_\varphi)J_\alpha f(z)$ on the universal covering over

$$\mathbb{C} \setminus \{-e^{-\varphi}, -e^{\varphi}\}.$$

It has the following behavior near the ramification points:

1. Near ∞ the function $F(z)$ has form $z^{-\alpha}\gamma(1/z)$, where γ is holomorphic near 0.

2. Near e^φ we have $F(z) = O(z - e^\varphi)^R$.

3. Near $e^{-\varphi}$ we have $F(z) = O(z - e^{-\varphi})^{R-\alpha}$.

Dominants $O(\cdot)$ are uniform in all sectors with finite central angles.

Next, we examine the function $g(t)$ given by (5.19). The function $F(z)z^{\alpha/2}$ is holomorphic in the sector $|\arg z| < \pi$ and admit estimates $O(|z|^{\alpha/2})$ at zero and $O(|z|^{-\alpha/2})$ at ∞. Therefore (see [31], Theorem 31), its Mellin transform $g(t)$ is

— holomorphic in the strip $|\operatorname{Im} t| < \alpha/2$,

— decreases as $O(e^{-(\pi-\varepsilon)|\operatorname{Re} t|})$ as $\operatorname{Re} t \to \pm\infty$.

Both consequences are not sufficient for our purposes[5]. For this reason, we improve a behavior of $F(z)$ at zero and at infinity (in the spirit of Watson's Lemma[6]).

Consider the functions

$$\tau_1(z) = \exp(-z^{1/3})\left(1 + z^{1/3} + \frac{1}{2!}z^{2/3}\right),$$

$$\tau_2(z) = z^{-\alpha}\exp(-z^{-1/3})\left(1 + z^{-1/3} + \frac{1}{2!}z^{-2/3}\right).$$

---

[5] If $\alpha \leqslant 2$, then the width of the strip is not sufficient.

[6] See, e.g., [1], Theorem C.3.1

Lemma 5.7 *The functions*

$$R(t) := \int_0^\infty \tau_1(z) z^{\alpha/2+it-1} dz, \qquad Q(t) := \int_0^\infty \tau_2(z) z^{\alpha/2+it-1} dz$$

are meromorphic in the strip

$$-\alpha/2 - 1 < \operatorname{Im} t < \alpha/2 + 1.$$

A unique singularity of $R(t)$ in the strip is a simple pole at $t = i\alpha$. A unique singularity of $Q(t)$ in the strip is a simple pole at $t = -i\alpha$. Both functions admit the following estimate in the strip

$$O(|t|^{3\alpha/2-1/2} e^{-3\pi|t|/2}), \qquad \operatorname{Re} t \to \pm\infty. \tag{5.22}$$

PROOF.

$$R(t) = 3\left(\Gamma(3\alpha/2 + 3it) + \Gamma(3\alpha/2 + 1 + 3it) + \frac{1}{2}\Gamma(3\alpha/2 + 2 + 3it)\right).$$

Poles of summands are $i\alpha/2$, $i\alpha/2 + i/3$, $i\alpha/2 + 2i/3$, but the last two poles cancel. □

Next, consider the function

$$F^\circ(z) := F(z) - F(0)\tau_1(z) - \left(z^\alpha F(z)\right)\Big|_{z=\infty} \cdot \tau_2(z).$$

Denote

$$g(t) = \int_0^\infty z^{\alpha/2} F(z) z^{is-1} ds, \qquad g^\circ(t) = \int_0^\infty z^{\alpha/2} F^\circ(z) z^{is-1} ds,$$

The function $z^{\alpha/2} F^\circ(z)$ admits the following expansions near 0 and ∞

$$z^{\alpha/2} F^\circ(z) = p_1 z^{\alpha/2+1} + p_2 z^{\alpha/2+2} + p_3 z^{\alpha/2+3} + \dots, \qquad |z| \to 0, \tag{5.23}$$

$$z^{\alpha/2} F^\circ(z) = q_1 z^{-\alpha/2-1} + q_2 z^{-\alpha/2-2} + q_3 z^{-\alpha/2-3} + \dots, \qquad |z| \to \infty \tag{5.24}$$

in the sector $|\arg z| \leqslant \pi$. It is continuous up to the boundary of the sector if $R > \alpha$. The functions

$$\gamma_\pm(x) := z^{\alpha/2} F^\circ(z)\Big|_{z=e^{x\pm i\pi}}$$

have $R - \alpha$ derivatives. Expansions (5.23)–(5.24) imply the following lemma.

Lemma 5.8 *All derivatives $\frac{d^k}{dx^k}\gamma_\pm(x)$ tend to zero as $x \to \pm\infty$.*

Therefore (see [31], Theorem 31 and proof of Theorem 26), $g^\circ(t)$ is holomorphic in the strip $|\operatorname{Im} t| < \alpha/2 + 1$ and satisfy the estimate

$$|g^\circ(t)| = O(e^{-\pi|\operatorname{Re} t|}|\operatorname{Re} t|^{-(R-\alpha)}), \qquad \operatorname{Re} t \to \pm\infty.$$

The function $g(t)$ satisfy the same estimate (because $g(t) - g^\circ(t)$ is (5.22)) at infinity, but it is meromorphic in the strip with simple poles at $t = \pm\alpha/2$.

Now it remains to divide[7] $g(t)$ by $\Gamma(\alpha/2 + it)\Gamma(\alpha/2 - it)$. Poles at $t = \pm i\alpha/2$ disappear, we get a function holomorphic in the strip $|\operatorname{Im} t| < 1 + \alpha/2$, and decreasing as $O(t^{-(R-2\alpha+1)})$. It remains to choose a sufficiently wide strip (5.21). $\qquad\square$

6   Example of self-adjoint extensions

This section contains another construction in Vilenkin's style, see [33], Section 7.7.11. A representation-theoretic standpoint of our considerations is explained at the end of the section.

6.1. The difference operator. Consider the space $L^2(\mathbb{R})$ and the subspace $\mathcal{V}$ consisting of functions holomorphic in the strip $|\operatorname{Im} s| \leqslant 1$ and decreasing as

$$|f(s)| = O(\operatorname{Re} s)^{-3/2-\varepsilon}, \qquad |\operatorname{Re} s| \to \infty.$$

Fix $\tau \in \mathbb{R}$, $0 < \varphi < \pi$ and consider the operator

$$\mathcal{L}f(s) = i(1/2 - is)f(s + i) + 2(s - \tau)\cos\varphi f(s) - i(1/2 + is - 2it)f(s - i).$$

It is symmetric, see Subsection 2.4.

Proposition 6.1 *The operator $\mathcal{L}$ is not self-adjoint. Its defect indices are* $(1, 1)$.

The author does not know are self-adjoint extensions of $\mathcal{L}$ natural objects or not. For this reason we consider another example.

Consider the operator $\mathcal{L} \oplus \mathcal{L}$ acting in the space $L^2(\mathbb{R}, ds) \oplus L^2(\mathbb{R}, e^{2\pi s}ds)$.

Consider the space $\mathcal{H}$ consisting of pair of functions (f_1, f_2) meromorphic in the strip $|\operatorname{Im} s| \leqslant 1$ such that

$$f_1(s) = O(\operatorname{Re} s)^{-3/2-\varepsilon}, \qquad |\operatorname{Re} s| \to \infty \tag{6.1}$$

$$e^{2\pi s}f_2(s) = O(\operatorname{Re} s)^{-3/2-\varepsilon}, \qquad |\operatorname{Re} s| \to \infty. \tag{6.2}$$

Fix $\sigma \in \mathbb{R}$. Consider the space $\mathcal{H}_\sigma$ consisting of pair of functions (f_1, f_2) meromorphic in the strip $|\operatorname{Im} s| \leqslant 1$ and satisfying (4.6), with simple poles at points $i/2$ and $-i/2 + 2\tau$. We also require

$$\operatorname*{res}_{s=i/2} f_1(s) = \operatorname*{res}_{s=i/2} f_2(s), \tag{6.3}$$

$$\operatorname*{res}_{s=-i/2+2\tau} f_1(s) = -e^{2\pi(\tau+i\sigma)} \operatorname*{res}_{s=-i/2+2\tau} f_2(s). \tag{6.4}$$

The parameter σ is present only in the last condition, it is a parameter of a self-adjoint extension.

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[7]The formula (5.16) contains also a multiplication by $e^{\pi t/2}$, but this factor cancels after (5.18).

Proposition 6.2 a) *The operator $\mathcal{L} \oplus \mathcal{L}$ has defect indices $(2,2)$ on $\mathcal{H}$.*

b) *The operator $\mathcal{L} \oplus \mathcal{L}$ is essentially self-adjoint on the domain $\mathcal{H}_\alpha$.*

Next, consider the following elements of the space $\mathcal{H}_\alpha$:

$$\left(\Psi_1^{(n)}(s), \Psi_2^{(n)}(s) \right), \tag{6.5}$$

where both functions $\Psi_1^{(n)}$, $\Psi_2^{(n)}$ are given by the same formula

$$B(1/2 + is, 1/2 + 2i\tau - is) \; {}_2F_1 \left[\begin{matrix} 1/2 + is, 1/2 - \sigma + i\tau - n \\ 1 + 2i\tau \end{matrix} ; 1 - e^{-2i\varphi} \right].$$

The function $\Psi_1^{(n)}$ is obtained by analytic continuation of

$$B(\ldots) \; {}_2F_1 \left[\begin{matrix} 1/2 + is, 1/2 - \sigma + i\tau - n \\ 1 + 2i\tau \end{matrix} ; z \right]$$

from $z = 0$ along the path $z = 1 - e^{-2i\theta}$ with $\theta \in [0, \varphi]$; $\Psi_2^{(n)}$ along the path $z = 1 - e^{2i\theta}$ with $\theta \in [0, \pi - \varphi]$.

Proposition 6.3 a) ([22]) *Elements $\left(\Psi_1^{(n)}(s), \Psi_2^{(n)}(s) \right)$, where n ranges in $\mathbb{Z}$, form an orthogonal basis in the space $L^2(\mathbb{R}, ds) \oplus L^2(\mathbb{R}, e^{2\pi s} ds)$.*

b) *They also are eigenfunctions of the operator $\mathcal{L} \oplus \mathcal{L}$ defined on $\mathcal{H}_\alpha$. The eigenvalues are $2 \sin \varphi (\sigma + n)$.*

6.2. A family of orthogonal bases in $L^2(\mathbb{R})$. Fix $\tau \in \mathbb{R}$, $\sigma \in \mathbb{C}$ and $\varphi \in (0, \pi)$. Define functions

$$\Delta_\sigma(x) = \Delta_\sigma(x; \tau, \varphi) = (1 + x e^{i\varphi})^{-1/2 - i\tau - \sigma} (1 + x e^{-i\varphi})^{-1/2 - i\tau + \sigma}.$$

We choose a branch of $\Delta_\sigma(x)$ by the condition $\Delta_\sigma(0) = 1$.

Lemma 6.4 *For any τ, $\sigma \in \mathbb{R}$, the functions $\Delta_{\sigma+n}$, where n ranges in $\mathbb{Z}$, form an orthogonal basis in $L^2(\mathbb{R})$.*

PROOF. We pass to a new variable $\theta \in [0, 2\pi]$ defined by

$$e^{i\theta} = \frac{1 + e^{i\varphi} x}{1 + e^{-i\varphi} x}, \qquad d\theta = \frac{2 \sin \varphi \, dx}{(1 + e^{i\varphi} x)(1 + e^{-i\varphi} x)}.$$

Then

$$(2 \sin \varphi)^{1/2 + i\tau} \cdot \Delta_{\sigma+n} = e^{-i(\sigma+n)\theta} \theta'(x)^{1/2 + i\tau}$$

We consider the map from $L^2[0, 2\pi]$ to $L^2(\mathbb{R})$ given by

$$S f(x) = f(\theta(x)) \theta'(x)^{1/2 + i\tau}$$

Evidently, it is unitary. The system $\Delta_{\sigma+n}$ is the image of the complete orthogonal system $e^{-i(\sigma+n)\theta}$ under the map S. □

6.3. A differential operator. Fix $\tau \in \mathbb{R}$, $\varphi \in (0,\pi)$. We consider the following symmetric differential operator

$$D = D_{\tau,\varphi} = i(x^2 + 2\cos\varphi x + 1)\frac{d}{dx} + i(1 + 2i\tau)(x + \cos\varphi) \qquad (6.6)$$

in $L^2(\mathbb{R}, dx/2\pi)$.

The functions Δ_σ are formal eigenfunctions of the operator D,

$$D\Delta_\sigma(x) = (2\sin\varphi)\sigma\Delta_\sigma(x). \qquad (6.7)$$

Lemma 6.5 a) *Defect indices of the operator D defined on the subspace $C_c^\infty(\mathbb{R})$ are $(1,1)$.*

b) *Defect indices of the operator D defined on the subspace $C_c^\infty\big((0,\infty)\big)$ are $(1,1)$.*

PROOF. Indeed, functions Δ_σ are contained in $L^2(\mathbb{R})$ for all $\sigma \in \mathbb{C}$. Therefore, $\dim\ker(D^* \pm i) = 1$. □

Fix $\sigma \in \mathbb{R}$. Denote by W_α the space of C^∞-functions on $\mathbb{R}$ such that there is a function $h(y)$ smooth near zero such that

$$f(x) = \begin{cases} x^{-1-2i\tau}h(1/x), & \text{for sufficiently large positive } x \\ e^{-2\pi i\sigma}(-x)^{-1-2i\tau}h(1/x), & \text{for sufficiently small negative } x \end{cases} \qquad (6.8)$$

Lemma 6.6 *The operator D is essentially self-adjoint on the subspace W_σ and $\Delta_{\sigma+n}$ are its eigenfunctions.*

PROOF. Verification of symmetry of D on W_σ is straightforward. The subspace W_σ contains vectors $\Delta_{\sigma+n}$. Other functions $\Delta_{\varkappa}$ are not in the domain of definiteness of D^* and therefore defect indices are $(0,0)$. □

6.4. The double Mellin transform. Let $f \in L^2(\mathbb{R})$. Consider the pair of functions

$$g_1(s) = \int_0^\infty f(x)x^{is-1/2}dx, \qquad (6.9)$$

$$g_2^\circ(s) = \int_{-\infty}^0 f(x)(-x)^{is-1/2}dx. \qquad (6.10)$$

Obviously,

$$\int_{-\infty}^\infty |f(x)|^2 dx = \frac{1}{2\pi}\left\{\int_{-\infty}^\infty |g_1(s)|^2 ds + \int_{-\infty}^\infty |g_2^\circ(s)|^2 ds\right\}.$$

Thus we get a unitary operator $L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, ds/2\pi) \oplus L^2(\mathbb{R}, ds/2\pi)$. Let modify this transform and set

$$g_2(s) = \int_{-\infty}^\infty f(x)x^{is-1/2}ds = -ie^{-\pi s}\int_0^\infty f(x)(-x)^{is-1/2}ds, \qquad (6.11)$$

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here we take a branch of $x^{is-1/2}$ that is analytic in the upper half-plane and real for $x > 0$. Now we get

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} |g_1(s)|^2 ds + \int_{-\infty}^{\infty} |g_2(s)|^2 e^{2\pi s} ds \right\}.$$

We denote the operator $f \mapsto (g_1, g_2)$ by $\widetilde{\mathfrak{M}}$

6.5. The difference operator. We evaluate the $\widetilde{\mathfrak{M}}$-image of Df as in (5.20) and get the formal difference operator $\mathcal{L} \oplus \mathcal{L}$ in $L^2(\mathbb{R}, ds) \oplus L^2(\mathbb{R}, e^{2\pi s} ds)$.

Propositions 6.1, 6.2.a are corollaries of the following lemma.

Lemma 6.7 a) *The image of $C_c^\infty(0, \infty)$ under (6.9) is contained in $\mathcal{V}$.*

b) *The image of $C_c^\infty(-\infty, 0) + C_c^\infty(0, \infty))$ is contained in $\mathcal{H}$.*

PROOF. a) Recall that the Mellin transform of f is reduced to the Fourier transform by the substitution $x = e^y$ to $f(x)$. In (6.9) we evaluate the Fourier transform of $f(e^y)e^{y/2}$, the function $g_1(s)$ decreases as $O(s^{-N})$ for any N.

b) We apply the same argument to $g_2^\circ(s)$, see (6.10). After passing to g_2 we get the estimate (6.2). □

Proposition 6.2.b is a corollary of the following lemma.

Lemma 6.8 *The image of the space W_α under the Mellin transform $\widetilde{\mathfrak{M}}$ is contained in the space $\mathcal{H}_\alpha$.*

PROOF. We repeat considerations in the spirit of Watson lemma. Pass to the function

$$f^\star(x) = \begin{cases} f(x) - f(0)e^{-x} - h(0)x^{-1-2i\tau}e^{-1/x}, & x > 0 \\ f(x) - f(0)e^x - h(0)e^{-2\pi i\sigma}(-x)^{-1-2i\tau}e^{1/x}, & x < 0 \end{cases}.$$

where h is the same as in (6.8). Consider the first component of the transform $\widetilde{\mathfrak{M}}$.

We have

$$f^\star(x) = c_1 x + \cdots + c_N x^N + O(x^{N+1}), \qquad\qquad x \to 0+ \quad (6.12)$$
$$f^\star(x) = d_1 x^{-2-2i\tau} + \cdots + d_M x^{-M-2i\tau} + O(x^{-M-1}), \qquad x \to +\infty. \quad (6.13)$$

Examine the behavior of

$$g_1(s) = \int_0^\infty f(x)x^{is-1/2} dx, \qquad g_1^\star(s) = \int_0^\infty f^\star(x)x^{is-1/2} dx.$$

Functions $g_1(s)$, $g_1^\star(s)$ are Fourier transforms of $f(e^y)e^{y/2}$, $f^\star(e^y)e^{y/2}$. It is easy to see that derivatives of $f^\star(e^y)e^{y/2}$ admit estimates

$$\frac{d^k}{dy^k}\left(f^\star(e^y)e^{y/2}\right) = O(e^{-3|y|/2}).$$

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Therefore $g_1^\star(s)$ is defined in the strip $|\operatorname{Im} s| < 3/2$ and decreases in this strip as $O(|\operatorname{Re} s|^{-N})$ for any N.

On the other hand,

$$g_1(s) - g_1^\star(s) = f(0)\Gamma(1/2 + is) + h(0)\Gamma(-1/2 - 2i\tau + is)$$

is meromorphic in the stir with poles at $s = i/2$, $s = -i/2 + 2i\tau$ and exponentially decreases as $|\operatorname{Re} s| \to \infty$. The residues at poles are $f(0)$ and $h(0)$ respectively.

In the same way we prove decreasing of $g_2^\circ(s)$ at infinity. Residues at poles $s = i/2$, $s = -i/2 + 2i\tau$ are respectively $f(0)$ and $e^{-2\pi\sigma i}$. It remains to multiply $g_2^\circ(s)$ by $-ie^{-\pi s}h(0)$ and we come to (6.3)–(6.4). $\qquad\square$

6.6. Proof of Proposition 6.3. We evaluate $\widetilde{\mathfrak{M}}\Delta_{\sigma+n}$ using the formula (5.17) and come to (6.5).

6.7. The origin of construction of this section. Fix σ, $\tau \in \mathbb{R}$. Consider the following representation $T_{\tau,\sigma}(g)$ of the group $\mathrm{SL}_2(\mathbb{R})$ in $L^2(\mathbb{R})$,

$$T_{\tau,\sigma}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = f\left(\frac{b+xd}{a+zc}\right)(a+zc)^{-1/2-\sigma+i\tau}\overline{\ln(a+zc)}^{-1/2+\sigma+i\tau}$$

In this formula, we choose any branch of $\ln(a + zc)$ that is holomorphic in the upper half-plane and define powers as

$$(a+zc)^{-1/2-\sigma+i\tau}\overline{(a+zc)}^{-1/2+\sigma+i\tau} :=$$
$$:= \exp\left((-1/2 - \sigma + i\tau)\ln(a+zc) + (-1/2 + \sigma + i\tau)\overline{(a+zc)}\right)$$

Thus, an operator $T_{\tau,\sigma}(g)$ is determined up to a constant factor and we get a projective unitary representation of $\mathrm{SL}(2,\mathbb{R})$ (it is a representation of the principal series, see, e.g., [24], Subsection 7.4.3).

The operator $D_{\tau,\varphi}$ given by (6.6) is an infinitesimal generator of the group $\mathrm{SL}_2(\mathbb{R})$. It generates a compact subgroup, and $\Delta_{\sigma+n}$ are eigenvectors of this subgroup.

The transform $\widetilde{\mathfrak{M}}$ is the spectral decomposition of the one-parametric group of operators $T_{\tau,\sigma}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

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